The Galerkin Method for the Equations Governing Formation of Salt Fingers: Implementation in *Mathematica*

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In this paper we consider the stability of solutions of the system of equations

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} & = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g} + \nu \Delta \mathbf{u}, \\ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T & = \kappa_T \Delta T, \\ \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S & = \kappa_S \Delta S. \end{cases} \tag{1}$$

Here $\mathbf{u} = \langle u(x,z,t), w(x,z,t)\rangle$ is the velocity field of a two dimensional motion, T and S are the temperature and salinity variables, ρ_0 is the reference density of the flow, κ_T and κ_S are the molecular diffusion coefficients of temperature and salt, p is pressure, $\mathbf{g}$ is the acceleration of gravity and ν is the viscosity. These equations are based on the Boussinesq approximation of the full equations of motion in which the density variation is only taken into account in the bouyancy term. We assume a linear constitutive equation that relates the denisty ρ to temperature and salinity:

$$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(S - S_0) \tag{2}$$

where ρ_0, T_0 and S_0 are reference (or mean) density, temperature and salinity, and α and β are coefficients of thermal and haline expansions. See References [1], [2], [3] for more information on the physical background of (1) and (2). In particular we assume that the velocity $\mathbf{v}$ is divergence free:

$$\text{div } \mathbf{u} = 0. \tag{3}$$

Since the flow is two-dimensional and $\mathbf{u}$ satisfies (3) we represent the velocity by a stream function $\psi(x,z,t)$ defined by

$$u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x}. \tag{4}$$

We can now eliminate p from (1) by taking the curl of the balance of linear momentum equation in (1) and use (2) and (4) to arrive at

$$\frac{\partial \Delta \psi}{\partial t} + J(\psi, \Delta\psi) = -g(\alpha T_x - \beta S_x) + \nu \Delta^2 \psi \tag{5}$$

where J is the jacobian operator. Similarly, the last two equations in (1) take the form

$$\begin{cases} \frac{\partial T}{\partial t} + J(\psi, T) & = -\psi_x + \kappa_T \Delta T, \\ \frac{\partial S}{\partial t} + J(\psi, S) & = \psi_x + \kappa_S \Delta S. \end{cases} \tag{6}$$

We begin our analysis (5)-(6) by noting that the uniform state $\mathbf{u} = 0$ and uniform gradient states of $T = T_0 \equiv az$ and $S = S_0 \equiv bz$ form a solution to (1), (2) and (3) if we choose $p = -\rho_0 gz$, the hydrostatic pressure, where $\mathbf{g} = -g\mathbf{k}$ with $\mathbf{k}$ the standard unit vector in the upward z-direction. Following [1] we denote the uniform gradients by Γ_T and $-\Gamma_S$ and consider solutions ψ, T and S that are deviations from the uniform state, that is, we define T' and S' so that

$$T = T' + \Gamma_T z, \quad S = S' + \Gamma_S z. \tag{7}$$

Because of the way T and S enter (5) and (6) quantities T' and S' simply replace T and S in these equations. So the following set of equations constitute the governing equations that we wish to study:

$$\left\{ \begin{array}{ll} \frac{\partial \Delta\psi}{\partial t} + J(\psi, \Delta\psi) & = -g(\alpha T'_x - \beta S'_x) + \nu\Delta^2\psi, \\ \frac{\partial T'}{\partial t} + J(\psi, T') & = -\psi_x + \kappa_T \Delta T', \\ \frac{\partial S'}{\partial t} + J(\psi, S') & = \psi_x + \kappa_S \Delta S'. \end{array} \right. \tag{8}$$

Our next task is to determine the non-dimensionalize euqations (9).

1 Non-dimensional Governing Equations

Following [1] we non-dimensionalize the above equations by the following transformation

$$x' = \frac{x}{d}, z' = \frac{z}{d}, t' = \frac{t}{\tau}, \psi' = \frac{\psi}{A}, T'' = \frac{T'}{,} S'' = \frac{S'}{s} \tag{9}$$

where

$$d = (\nu\kappa_T / g\alpha\Gamma_T)^{(1/4)}, \tau = d^2/\kappa_T, h = d\Gamma_T, s = d\Gamma_S. \tag{10}$$

We consider equations of motion

$$\left\{ \begin{array}{ll} \frac{\partial \Delta\psi}{\partial t} + J(\psi, \Delta\psi) & = \sigma T'_x - \frac{\sigma}{R} S'_x + \sigma\Delta^2\psi, \\ \frac{\partial T'}{\partial t} + J(\psi, T') & = -\psi_x + \Delta T', \\ \frac{\partial S'}{\partial t} + J(\psi, S') & = \psi_x + \tau \Delta S', \end{array} \right. \tag{11}$$

where ψ is the stream function and T' and S' are deviations from the steady-state solutions of temperature and salinity. We look for solutions of the form

$$\left\{ \begin{array}{ll} \psi(x,y,t) & = \sum_{n=1}^N \sum_{m=1}^N a_{mn}(t)\phi_{mn}(x,y), \\ T'(x,y,t) & = \sum_{n=1}^N \sum_{m=1}^N b_{mn}(t)\phi_{mn}(x,y), \\ S'(x,y,t) & = \sum_{n=1}^N \sum_{m=1}^N c_{mn}(t)\phi_{mn}(x,y), \end{array} \right. \tag{12}$$

where we choose

$$\phi_{mn}(x,y) = \sin nx \sin my, \tag{13}$$

as the basis in the Galerkin scheme. This choice is motivated by our desire to solve (11) in the domain $(0,\pi) \times (0,\pi)$ with zero boundary conditions. Next we construct the three operators T_1, T_2 and T_3 from the left-sides of (11), i.e.,

$$\left\{ \begin{array}{ll} T_1[\psi, T', S'] & = \frac{\partial \Delta\psi}{\partial t} + J(\psi, \Delta\psi) - \sigma T'_x + \frac{\sigma}{R} S'_x - \sigma\Delta^2\psi, \\ T_2[\psi, T', S'] & = \frac{\partial T'}{\partial t} + J(\psi, T') + \psi_x - \Delta T', \\ T_3[\psi, T', S'] & = \frac{\partial S'}{\partial t} + J(\psi, S') - \psi_x - \tau\Delta S'. \end{array} \right. \tag{14}$$

The final step of the Galerkin scheme is to take the inner product of (14) with a tyoical basis element in (13):

$$\begin{cases} (T_1[\psi, T', S'], \phi_{ij}) &= 0, \\ (T_2[\psi, T', S'], \phi_{ij}) &= 0, \quad \text{for } i, j = 1, ..., N, \\ (T_3[\psi, T', S'], \phi_{ij}) &= 0. \end{cases} \tag{15}$$

The simplest approximation to the solution to (11) is obtained when we let $N = 1$ in (15). In that case the solution triple (ψ, T', S') is approximated by

$$\begin{cases} \psi(x, y, t) &= a_{11}(t) \sin x \sin y, \\ T'(x, y, t) &= b_{11}(t) \sin x \sin y, \\ S'(x, y, t) &= c_{11}(t) \sin x \sin y. \end{cases} \tag{16}$$

In equations (16) only the very first mode (the fundamental mode) is taken into consideration. In this special case (15) reduce to the following set of uncoupled linear equations for the unknowns a_{11}, b_{11} and c_{11}:

$$\begin{cases} a'_{11} &= -2\sigma a_{11}, \\ b'_{11} &= -2b_{11}, \\ c'_{11} &= -2\tau c_{11}. \end{cases} \tag{17}$$

These equations show that the only equilibrium solution $a_{11} = 0$, $b_{11} = 0$ and $c_{11} = 0$ is stable as long as $\tau > 0$. In other words, the equilibrium solution of (11), given by $\psi = 0$ (no motion) and $T' = S' = 0$ (corresponsing to homogeneous distribuation of temperature and salinity) is linearly stable as long as $\tau > 0$.

By contrast we set $N = 2$ our approximation takes the form

$$\begin{cases} \psi(x, y, t) &= a_{11}(t) \sin x \sin y + a_{12}(t) \sin x \sin 2y + a_{21}(t) \sin 2x \sin y + a_{22}(t) \sin x \sin y, \\ T'(x, y, t) &= b_{11}(t) \sin x \sin y + b_{12}(t) \sin x \sin 2y + b_{21}(t) \sin 2x \sin y + b_{22}(t) \sin x \sin y, \\ S'(x, y, t) &= c_{11}(t) \sin x \sin y + c_{12}(t) \sin x \sin 2y + c_{21}(t) \sin 2x \sin y + c_{22}(t) \sin x \sin y. \end{cases} \tag{18}$$

The representations in (18) are already considerably richer than (16) because they take into account the interactions between two modes. The equations in (17) now take the form

$$\begin{cases} a'_{11} &= -2\sigma a_{11} + \frac{4\sigma}{3\pi} b_{21} - \frac{4\sigma}{3\pi R} c_{21}, \\ b'_{11} &= \frac{8}{3\pi} a_{21} - 2b_{11} - \frac{3}{4} a_{21} b_{12} + \frac{3}{4} a_{12} b_{21}, \\ c'_{11} &= -\frac{8}{3\pi} a_{21} - 2\tau c_{11} - \frac{3}{4} a_{21} c_{12} + \frac{3}{4} a_{21} c_{12}, \\ a'_{12} &= -5\sigma a_{12} - \frac{9}{20} a_{11} a_{21} + \frac{8\sigma}{15\pi} b_{22} - \frac{8\sigma}{15\pi R} c_{22}, \\ b'_{12} &= \frac{8}{3\pi} a_{22} + \frac{3}{4} a_{21} b_{11} - 5b_{12} - \frac{3}{4} a_{11} b_{21}, \\ c'_{12} &= \frac{-8}{3\pi} a_{22} + \frac{3}{4} a_{21} c_{11} - 5\tau c_{12} - \frac{3}{4} a_{11} c_{21}, \\ c'_{21} &= \frac{8}{3\pi} a_{11} - \frac{3}{4} a_{12} c_{11} + \frac{3}{4} a_{11} c_{12} - 5\tau c_{21}, \\ a'_{21} &= \frac{9}{20} a_{11} a_{12} - 5\sigma a_{21} - \frac{8\sigma}{15\pi} b_{11} + \frac{8\sigma}{15\pi R} c_{11}, \\ b'_{21} &= \frac{-8}{3\pi} a_{11} - \frac{3}{4} a_{12} b_{11} + \frac{3}{4} a_{11} b_{12} - 5b_{21}, \\ a'_{22} &= -8\sigma a_{22} - \frac{\sigma}{3\pi} b_{12} + \frac{\sigma}{3\pi} c_{12}, \\ c'_{22} &= \frac{8}{3\pi} a_{12} - 8\tau c_{22}, \\ b'_{22} &= -\frac{8}{3\pi} a_{12} - 8b_{22} \end{cases} \tag{19}$$

Note that equations (19) are not only coupled, they are nonlinear. Moreover, unlike the equations in (17), (19) show the interplay among all of the physical parameters R, σ and τ.

Our goal now is to study the behavior of equilibrium solutions of (19) in terms of the three physical parameters. In particular, we would be very interested in locating values of R, σ and τ for which the zero (equilibrium) solution in (18) is unstable. It is our belief that, analogous to Rayleigh's analysis of the Boussinesq equations which led to the discovery of the celebrated Rayleigh-Bénard cells, when the zero solution is unstable, the resulting stable solution will have features that are common with salt fingers.

2 References

[1] Merryfield, W., J., G. Holloway, and A. E. Gargett, Differential vertical transport of heat and salt by weak stratified turbulence, Amer. Geophys. U., 1998, pp. 2773 – 2776.
[2] Howard, L. N. and G. Veronis, Stability of salt fingers with negligible diffusivity, J. Fluid Mech. (1992), vol 239, pp. 511 – 522.
[3] Stern, M. E., Collective instability of salt fingers, J. Fluid Mech. (1969), vol 35, pp. 209 –